Order **4** (1987), 155–164.

Regressions and Monotone Chains II: The Poset of Integer Intervals

NOGA ALON
Tel Aviv University, Tel Aviv, Israel

W. T. TROTTER
Arizona State University, Tempe, AZ 85287, U.S.A

and

DOUGLAS B. WEST
University of Illinois, Urbana, IL 61801, U.S.A.

Communicated by P. Hell

(Received: 24 September 1986; accepted: 4 March 1987)

Abstract. A *regressive function* (also called a *regression* or *contractive mapping*) on a partial order P is a function σ mapping P to itself such that $\sigma(x) \leqslant x$. A *monotone k-chain* for σ is a k-chain on which σ is order-preserving; i.e., a chain $x_1 < \cdots < x_k$ such that $\sigma(x_1) \leqslant \cdots \leqslant \sigma(x_k)$. Let P_n be the poset of integer intervals $\{i, i+1, \ldots, m\}$ contained in $\{1, 2, \ldots, n\}$, ordered by inclusion. Let $f(k)$ be the least value of n such that every regression on P_n has a monotone $k+1$-chain, let $t(x, j)$ be defined by $t(x, 0) = 1$ and $t(x, j) = x^{t(x, j-1)}$. Then $f(k)$ exists for all k (originally proved by D. White), and $t(2, k) < f(k) < t(e + \varepsilon_k, k)$, where $\varepsilon_k \to 0$ as $k \to \infty$. Alternatively, the largest k such that every regression on P_n is guaranteed to have a monotone k-chain lies between $\lg^*(n)$ and $\lg^*(n) - 2$, inclusive, where $\lg^*(n)$ is the number of applications of logarithm base 2 required to reduce n to a negative number. Analogous results hold for *choice functions*, which are regressions in which every element is mapped to a minimal element.

AMS subject classifications (1980). 05C55, 06A10, 62J.

Key words. Regressions, Ramsey theory.

1. Introduction

A *regressive function* (also called a *regression* or *contractive mapping*) on a partial order P is a function σ mapping P to itself such that $\sigma(x) \leqslant x$. We consider conditions on P that force any regression on P to have a long chain on which it is order-preserving; i.e., a chain $x_1 < \cdots x_k$ such that $\sigma(x_1) \leqslant \cdots \leqslant \sigma(x_k)$. Such a chain is called a *monotone k-chain*. In discussing lattices, we delete the unique minimal element from consideration.

This problem is Ramsey-theoretic, in that an arbitrary regression on P is analogous to a coloring, and we ask for a sub-coloring having additional structure. Hence it is not surprising that for various classes of posets, sufficient size forces a monotone k-chain in any regression. For example, in [10] we showed that this holds for any poset of width w that has at least $(w+1)^{k-1}$ elements.

The question has also been studied for Boolean algebras and other well-behaved families of lattices. The arguments that have guaranteed monotone chains in these well-behaved families in fact guaranteed monotone chains with special additional properties. In the terminology of Harzheim [2], they guarantee a *fix chain* ($\sigma(x_i)=x_i$ for $1 \leqslant i \leqslant k$), a *constant chain* ($\sigma(x_i)=y$ for $1 \leqslant i \leqslant k$), or a *regressive chain* ($\sigma(x_i)=x_{i-1}$ for $2 \leqslant i \leqslant k$). Voigt [9] defined a class $\mathbf{L}$ of ranked posets to have the *regressive chain property* if for every k there exists n such that every regression on every poset in $\mathbf{L}$ with rank at least n has a fix, monotone, or constant chain. Harzheim [2, 3] proved that Boolean algebras have the regressive chain property, generalizing results of Harzheim [1], Rado [8], and Leeb [6]. Voigt [9] proved a further generalization to classes $\mathbf{L}$ satisfying a Ramsey-theoretic property. As applications, he showed that Boolean algebras, partition lattices, and subspace lattices have the regressive chain property.

Authors have also considered a restricted class of regressions, called *choice functions*, in which $\sigma(x)$ is a minimal element for all $x \in P$. Note that a monotone chain for a choice function is a constant chain. It is natural to ask how large must n be to guarantee monotone k-chains for arbitrary regressions or choice functions. Most of the results mentioned above are non-constructive and give no such bounds. However, the exact value is known for choice functions on Boolean algebras. Perry [7] proved that the minimum n such that every choice function on the non-empty subsets of $\{1, ..., n\}$ has a constant k-chain is $n=2^{k-1}$. (Harzheim [1] gave the lower bound, and Kleitman and Lewin [4] gave another proof.)

In this paper we study regressions on special subposets of the Boolean algebras. Let P_n be the poset of integer intervals $\{i, i+1, ..., j\}$ contained in $\{1, ..., n\}$, ordered by inclusion; $|x|$ denotes the size of an interval $x \in P_n$. Let $f(k)$ be the least value of n such that every regression on P_n has a monotone $k+1$-chain. We show that $t(2,k)<f(k)<t(e+\varepsilon_k, k)$, where $\varepsilon_k \to 0$ and $t(x,j)$ is defined by $t(x,0)=1$ and $t(x,j)=x^{t(x,j-1)}$. More precisely, $f(k)<t(e+\varepsilon, k)$ for any fixed $\varepsilon>0$ as long as k is sufficiently large. In fact, since $f(1)=2$ and $f(2)=8$, it is likely that $f(k)<t(e+\varepsilon_k, k)$ with $\varepsilon_k<0$ for all $k>0$, but we prove only that there exists a decreasing sequence ε_k with $\varepsilon_1=1$, for example. We refer to such a fixed sequence henceforth whenever we write $t(e+\varepsilon_k, k)$. Our actual upper bound is exponentially recursive, and $t(e+\varepsilon_k, k)$ is an upper bound on the solution to the recurrence. As in the earlier investigations, we can guarantee monotone k-chains of a special form. An *interlaced monotone k-chain* for σ is a monotone k-chain $x_1<\cdots<x_k$ such that for every $i<k$ either $\sigma(x_{i+1}) \geqslant x_i$ or $\sigma(x_{i+1})=\sigma(x_i)$. For $n \geqslant t(e+\varepsilon_k, k)$, every regression on P_n has an interlaced monotone $k+1$-chain, but there is a regression on $P_{t(2,k)}$ with no monotone $k+1$-chain of any kind.

Furthermore, let $g(k)$ be the least value of n such that every choice function on P_n has a monotone $k+1$-chain. We show that $2^k \leqslant g(k) < e^{k+423}$. The existence of $f(k)$ and $g(k)$ was originally proved by D. White [12] in proving Milner's conjecture that regressions on the poset of all integer intervals have arbitrarily long monotone chains. His upper bound on $g(k)$ is almost the same as ours, obtained before finding his paper. We include our proof because the argument generalizes to give a new upper bound on $f(k)$. Our lower bounds are new and obtained by explicit constructions. They are not best possible, since they yield $g(2) \geqslant 4$ and $f(2) \geqslant 6$, but $g(2) = 5$ and $f(2) = 8$.

To compare our bound on $g(k)$ with [12], let $H(n) = \sum_{i=1}^{n} 1/i$. White showed that $H(n) \geqslant (k+1)n/(n+1)$ guarantees that every regression on P_n has a monotone k-chain. We show that $H(n) > (k+1)n/(n+1)$ guarantees that every regression on P_n has a monotone $k+1$-chain – a very slight improvement. It is well-known (see [5, p. 74]) that

$$H(n) = \ln n + \gamma + 1/2n - 1/12n^2 + \cdots > \ln n + \gamma,$$

where $\gamma = 0.57721\ldots$ is Euler's constant. Hence it suffices to have $\ln n > k + 1 - \gamma$, so that $g(k) < e^{k+423}$.

There is another way to interpret the bounds on $f(k)$. We always have ε_k small enough so that $f(k) < 4t(4, k)$. A short proof by induction shows that $4t(4, k) \leqslant t(2, k+2)$. (Equality holds for $k = 0$, and

$$\lg 4t(4, k) = 2 + 2t(4, k-1) \leqslant 4t(4, k-1) \leqslant t(2, k+1) = \lg t(2, k+2).)$$

This allows us to answer, within an additive constant, the question of how large a monotone chain is forced in every regression on a particular P_n. The answer is between $\lg^*(n)$ and $\lg^*(n) - 2$, where $\lg^*(n)$ is the number of times the logarithm base 2 must be iterated to reduce n to a negative number.

We present the upper bounds in Section 2 and lower bounds in Section 3. In Section 4, we mention several straightforward generalizations.

2. Upper Bounds

First we show that an upper bound on $f(k)$ can be obtained from an upper bound on the function g. Then we obtain an upper bound for g. Finally the method used to do that extends to yield an upper bound for f directly. The upper bounds are obtained recursively and thus depend on the existence of $f(0)$ and $g(0)$, which both equal 1. The ideas in this lemma are similar to ideas used by White in proving Milner's conjecture from the existence of g.

LEMMA 1. *Given the existence of the function* g, $f(k) \leqslant g(1 + kf(k-1))$.

Proof. For $k \geqslant 1$, consider any arbitrary regression σ on P_n, where $n \geqslant g(1 + kf(k-1))$. Define a choice function τ on P_n by letting $\tau(x)$ be the largest value in $\sigma(x)$ (i.e., its right endpoint). By the choice of n, τ is constant on a chain C of at least $1+kf(k-1)$ elements. The images $Y=\{\sigma(x): x\in C\}$ lie on a single chain, since they are intervals with the same right endpoint. If Y has at most $f(k-1)$ distinct images, then the pigeonhole principle says that σ is constant on a subchain C' with more than k elements. If $|Y|$ is larger, then we have some $\sigma(x)=y$ where y has size $m>f(k-1)$. The subintervals of y form a copy of P_m, and σ restricts to a regression on it. This restriction must have a monotone k-chain, and x can be added to the top to obtain a monotone $k+1$-chain under σ. □

The upper bound on $g(k)$ is obtained by weighting the elements of P_n in such a way that the total weight grows faster than the number of minimal elements, and such that large weight mapped to a single element implies its pre-image contains a $k+1$-chain. As noted in the introduction, this theorem implies $g(k)<e^{k+423}$.

THEOREM 1. *If $H(n)>(k+1)n/(n+1)$, then every choice function on P_n has a monotone $k+1$-chain.*

Proof. Assign to each element of size i in P_n a weight of $1/i$. Since P_n has $n+1-i$ elements of size i, the total weight in P_n is $(n+1)H(n)-n$, which under the hypothesis exceeds kn. Any choice function maps all of this weight to the n minimal elements, so some element y receives total weight exceeding k. To show that $\sigma^{-1}(y)$ contains a $k+1$-chain, we need only show that any antichain in $\sigma^{-1}(y)$ has total weight at most 1.

Let $U=\{x\in P_n: x\geqslant y\}$; note $\sigma^{-1}(y)\subseteq U$. Furthermore, $U\subset Q$, where Q is the product of two chains. Extend the weighting to Q by giving weight $1/i$ to all elements at the ith level (the minimal element y has weight 1). In Q, any antichain restricted to a single rank has total weight 1. For any other antichain in Q, replacing the elements appearing at the lowest level by those that directly cover them yields an antichain of larger weight, because the j elements of weight $1/i$ are replaced by at least $j+1$ elements of weight $1/(i+1)$, and $(j+1)/(i+1)>j/i$ when $j<i$. Any antichain in Q can be pushed up to a single rank by this procedure, hence antichains in Q (and their restrictions to U) have weight at most 1. □

Using Lemma 1, this theorem yields the recursive upper bound $f(k)\leqslant e^{1.423+kf(k-1)}$. A close look at the proof of Lemma 1 suggests that it can be improved. Indeed, by applying the weighting technique directly to the regression problem, we can replace the $kf(k-1)$ by $\sum_{i=0}^{k-1} f(i)$, which is much smaller. In addition, the monotone $k+1$-chain that we force when n is of the size has the nice structure defined in the introduction. The exponentiation causes this bound to grow so quickly that we can ignore the sum if we slightly increase the base.

THEOREM 2. *If*

$$H(n) > \frac{1}{n+1}\, n\left(1+\sum_{i=0}^{k-1} f(i)\right) - \frac{1}{n+1}\sum_{i=0}^{k-1}\binom{f(i)}{2},$$

then any regression σ *on* P_n *has an interlaced monotone* $k+1$*-chain. In particular,* $f(k) < e^{423+\sum_{i<k} f(i)} < t(e+\varepsilon_k, k)$.

Proof. Suppose $|y| \geq f(i)$ and $\sigma^{-1}(y)$ contains a $k-i$-chain of elements larger than y. This chain can be placed at the top of the interlaced monotone $i+1$-chain guaranteed in the copy of $P_{|y|}$ generated by y to obtain a monotone $k+1$-chain for σ. This $k+1$-chain will be interlaced, since it will satisfy $\sigma(x_{k+1}) = \cdots = \sigma(x_{i+2}) \geq x_{i+1}$, and the interlacing condition is satisfied by hypothesis for $j \leq i$.

With the weight function used earlier, again the weight of any antichain in $\sigma^{-1}(y)$ is at most 1. (In fact, the weight is at most $(n-|y|+2)/(n+|y|)$, and this is attained only by intervals centered at $n/2$; to further improve the bound, subtract 1 from numerator and denominator for each unit of displacement of the interval from that center.) Avoiding an interlaced monotone $k+1$-chain thus limits the total weight in P_n. In particular, every y with $f(i-1) < |y| \leq f(i)$ absorbs weight at most $k-i$. Equivalently, letting N_i be the number of elements $y \in P_n$ with $|y| \leq f(i)$, the bound on allowable weight is $\sum_{i=1}^{k} N_{k-i}$. To evaluate this sum, note that

$$N_i = nf(i) - \binom{f(i)}{2}.$$

Hence when the total weight reaches $n\sum_{i=0}^{k-1} f(i) - \sum_{i=0}^{k-1}\binom{f(i)}{2}$, an interlaced monotone $k+1$-chain is forced. As before, the total weight in P_n is $(n+1)H(n) - n$.

Replacing the upper bound by $t(e+\varepsilon_k, k)$ is done inductively. We have

$$f(k) \leq e^{423+\sum_{i<k} f(i)} < e^{423+\sum_{i<k} t(e+\varepsilon_i, i)} < t(e+\varepsilon_k, k). \qquad \square$$

3. Lower Bounds

To describe the constructions used to obtain lower bounds, it is convenient to encode the elements of P_n by their smallest element and size; $\{i, \ldots, i+j-1\}$ will be denoted (i, j). First we consider the problem of choice functions, where we write $\sigma(i,j) = r$ rather than $\sigma(i,j) = (r, 1)$.

THEOREM 3. $g(k) \geq 2^k$.

Proof. We construct a choice function on P_{2^k-1} with no monotone $k+1$-chain. We must define σ so that the preimage of each minimal element y is the union of k antichains (y itself must be one of these). To this end we partition P_{2^k-1} into k bands of consecutive complete levels and define σ so that the elements of a single band that map to y form an antichain.

Index the levels of P_n by the size of the elements as integer intervals. Let the ith band consist of levels 2^{i-1} through 2^i-1, for $1 \leqslant i \leqslant k$. For $r>1$, let $\sigma(r,s)=\sigma(r-1,s)+1$. If the images of $\{(1,s)\}$ are legally defined, this will yield a well-defined choice function on all of P_{2^k-1}. In the ith band, we must define σ for $(1,2^{i-1}), (1,2^{i-1}+1), \ldots, (1,2^i-1)$. Let the images, respectively, be the singletons $2^{i-1}, 2^{i-1}-1, \ldots, 1$. Since $2^{i-1} \in (1, 2^{i-1})$, this defines a legal choice function; note that the first band is the bottom level, and these elements are fixed under σ. In general, the elements of the ith band that map to t are $\{(t+j-2^{i-1}, 2^{i-1}+j-1): 1 \leqslant j \leqslant 2^{i-1}\}$, where some of these do not eist if $t<2^{i-1}$ or $t>n-2^i+2$. The construction works because this collection forms an antichain. This choice function is illustrated on the left in Figure 1 by labeling each point with the element chosen by σ. □

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4 5 6 7 8 9 ...
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3,1 4,1 5,1 6,1 7,1 8,1 9,1 ...
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1,2 2,2 3,2 4,2 5,2 6,2 7,2 8,2 ...
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1,1 2,1 3,1 4,1 5,1 6,1 7,1 8,1 9,1 ...

Fig. 1. A choice function and regression without long monotone chains.

The idea behind this construction also yields a construction for a general regression. It gives a lower bound on $f(k)$ that is stronger than $f(k) > t(2, k)$.

THEOREM 4. *$f(k) \geqslant N_k$, where $N_1 = 2$ and $N_k = 2^{1+N_{k-1}} - 2$.*

Proof. We define a regression σ on $P_{N_k - 1}$ using k bands of levels, in such a way that two elements in a single band can never belong to the same monotone chain. This requires that comparable elements in a single band have images that are incomparable or related in the opposite order. We specify the images by the ordered pair (initial element, size). Again we put $\sigma(r, s) = (t + 1, q)$ if $\sigma(r-1, s) = (t, q)$, so it suffices to specify $\sigma(1, s)$. As we develop the construction the reader may wish to refer to Fig. 1, where σ is described by labeling elements by their images.

In this construction each band consists of sub-bands of consecutive levels. (In Figure 1 sub-bands are separated by dotted lines, bands by solid lines.) Within a sub-band the size (level) of every image is the same. The image level for two sub-bands of a single band is in reverse order to their positions. This insures that no monotone chain contains a point of each; their images are incomparable or in reverse order. Within a single sub-band, the distinct images are all incomparable, since they lie at the same level, and we make the collection of elements mapping to a single element an antichain by the same sort of construction used in the previous proof. Hence a monotone chain uses at most one element from each band. It remains to determine how many levels this construction accommodates.

The lowest sub-band of each band is a single level consisting of fixed-points under σ. If this is level q, the jth sub-band of this band consists of elements mapping to level $q+1-j$. To make the sub-bands large, map the first element to the right-most possible element in its destination level. For example, if a sub-band starts at level q and maps to level r, put $\sigma(1, q) = (q - r + 1, r)$. The full sub-band is described by letting the images of $(1, q), (1, q+1), \ldots, (1, 2q - r)$ be $(q - r + 1, r), (q - r, r), \ldots, (1, r)$. In particular, it has $q - r + 1$ levels.

This enables us to compute the width of a band from its starting level q_1. The first sub-band starts at level q_1 and maps to level $r_1 = q_1$. There are q_1 sub-bands, each successive one mapping to a lower level. For the jth sub-band, $r_j = r_{j-1} - 1$, and $q_j = q_{j-1} + (q_{j-1} - r_{j-1} + 1)$. Thus $q_j - r_j + 1 = 2(q_{j-1} - r_{j-1} + 1) + 1$; i.e., the number of levels in the jth sub-band is one more than twice that in the previous sub-band. Since the first sub-band has 1 level, the jth has $2^j - 1$ levels, and the full band has $2^{1+q_1} - 2 - q_1$ levels.

To show that the first k bands cover $N_k - 1$ levels, we need only show the first level of band $k+1$ will be N_k. This is true for $k = 1$. Inductively, suppose the first level of band k is N_{k-1}. By the discussion above, band k has N_{k-1} sub-bands and $2^{1+N_{k-1}} - 2 - N_{k-1}$ levels. Adding these to N_{k-1} puts the first level of band $k+1$ at N_k. □

4. Other Remarks

The counting technique used in Section 2 can be applied to other families of posets in which every principle ideal is also a member of the family. We seek a weight function satisfying two properties. (1) The weights must be small enough to get bounds on the total weight of antichains (or unions of antichains) in $\{x : x \geqslant y\}$ for any $y \in P$. (2) The weights must be large enough so that the total weight grows 'quickly enough' with the size of the poset. As in Theorem 2, this will yield interlaced monotone k-chains for large enough members of the family. The first example is very similar to P_n, constructed so that each element covers $m+1$ elements rather than 2.

THEOREM 5. *Suppose $n = rm$, and let $P_{n,m}$ be the subposet of P_n induced by $\{x \in P_n : m \text{ divides } |x|\}$. Fix m and let n (and r) grow. If*

$$H(r) > m[(rm - m + 1)k + r]/(rm + 1),$$

then any choice function on $P_{n,m}$ has a constant $k+1$-chain.

Proof. Give each element x the weight it received in P_n, i.e., $1/|x|$. Then the antichains can be viewed as antichains in P_n and have the same weight bound. $P_{rm,m}$ has r levels; sum the weight in each to get total weight

$$\sum_{i=1}^{r} (n - mi + 1)/mi = (n+1)H(r)/m - r.$$

There are $(n - m + 1)$ minimal elements, so

$$H(r) > m[(rm - m + 1)k + r]/(rm + 1)$$

guarantees a constant $k+1$-chain for any choice function on $P_{rm,m}$. Note that $H(r)$ grows and the required value does not. □

Similarly, the argument of Theorem 2 can be followed to force interlaced monotone $k+1$-chains in $P_{rm,m}$ when r is sufficiently large.

There are other generalizations of P_n worth considering, particularly higher-dimensional analogues. For example, P_n can be viewed as the top n levels of the Cartesian product of two n-chains. Let P_n^t be the top n levels of the Cartesian product of $t+1$ n-chains. Then level i has $\binom{n-i+t}{t}$ elements (this is the number of nonnegative t-vectors with total sum $n-i$). For any element $y \in P_n^t$, $\{x : x \geqslant y\}$ is a product of $t+1$ chains of size at most n. The antichain weight in preimages can be bounded by 1 by choosing weights equal to $\binom{i-1+t}{t}^{-1}$ for elements in level i. An ad hoc pushing argument can be made for this as for P_n, but it also follows immediately from the fact that the product of $t+1$ chains is an LYM order (see [0] or [11] for definition and discussion).

In particular, consider the total weight in P_n^2. This equals

$$\sum_{i=1}^{n} \binom{n-i+2}{2} \Big/ \binom{i+1}{2} = (n+2)^2 \sum_{i=1}^{n} \frac{1}{i(i+1)} - (n+2)[H(n+1)-1]+n,$$

which grows quadratically since

$$\sum_{i\geqslant 1} \frac{1}{i(i+1)}$$

converges. Unfortunately, the number of elements on level 1 is $\binom{n-1}{2}$, which also grows quadratically. Hence this simple form of the argument cannot be used to show that sufficiently large n will force arbitrarily long constant chains for choice functions on P_n^2. It is possible that here the more accurate counting of weights or tighter bounds on the antichain weight in the elements above minimal elements far from the 'center' will suffice to show this.

Other generalizations to consider include Cartesian products of t copies of P_n and the inclusion ordering on unions of up to t integer intervals. The latter are very closely related to P_n^{2t}. The former include P_n as a principle ideal, so large enough values of n force long monotone chains; the problem is to discover by how much n can be reduced in comparison to $f(k)$ and $g(k)$ to force this in the product of t copies of P_n. Finally, we note that this technique will give an upper bound on n such that any choice function on $B_n - \{\emptyset\}$ is constant on some $k+1$-chain, but it will not give the best-possible bound in Perry's Theorem [4,7].

Acknowledgement

We thank Berndt Voigt for suggesting this problem.

References

0. C. Greene and D. J. Kleitman (1976) Proof techniques in the theory of finite sets, in *Studies in Combinatorics* (G. -C. Rota, ed.) MAA Studies in Mathematics, Vol. 17.
1. E. Harzheim (1968) Ein kombinatorisches Problem über Auswahlfunktionen, *Publ. Math. Debrecen* **15**, 19–22.
2. E. Harzheim (1982) Combinatorial theorems on contractive mappings in power sets, *Discrete Math.* **40**, 193–201.
3. E. Harzheim (1983) A constructive proof for a theorem on contractive mappings in power sets, *Discrete Math.* **45**, 99–106.
4. D. J. Kleitman and M. Lewin (1972) Another proof of a result of Perry on chains of finite sets, *Amer. Math. Monthly* **79**, 152–154.
5. D. E. Knuth (1968) *The Art of Computer Programming, Vol. I.* Addison-Wesley.
6. K. Leeb (1973) Vorlesungen über Pascaltheorie, Erlangen.
7. R. L. Perry (1967) Representatives of subsets, *J. Comb. Theory* **3**, 302–304.
8. R. Rado (1971) A theorem on chains of finite sets, II, *Acta Arith.* **43**, 257–261.
9. B. Voigt (1983) Combinatorial properties of regressive mappings, *Discrete Math.* **47**, 97–108.
10. D. B. West, W. T. Trotter, G. W. Peck, and P. Shor (1985) Regressions and monotone chains, *Combinatorica*.

11. D. B. West (1982) Extremal problems in partially ordered sets, in *Ordered Sets* (I. Rival, ed.), D. Reidel, Dordrecht, pp. 473–521.
12. D. J. White (1967) On finite nests of intervals of integers, *J. London Math. Soc.* **42**, 501–503.